

# HODGE LOCUS AND BRILL-NOETHER TYPE LOCUS

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**ABSTRACT.** Given a family  $\pi : \mathcal{X} \rightarrow B$  of smooth projective varieties, a closed fiber  $\mathcal{X}_o$  and an invertible sheaf  $\mathcal{L}$  on  $\mathcal{X}_o$ , we compare the Hodge locus in  $B$  corresponding to the Hodge class  $c_1(\mathcal{L})$  with the locus of points  $b \in B$  such that  $\mathcal{L}$  deforms to an invertible sheaf  $\mathcal{L}_b$  on  $\mathcal{X}_b$  with at least  $h^0(\mathcal{L})$ -dimensional space of global sections (it is a Brill-Noether type locus associated to  $\mathcal{L}$ ). We finally give an application by comparing the Brill-Noether locus to a family of curves on a surface passing through a fixed set of points.

## 1. INTRODUCTION

The base field  $k$  is always assumed to be algebraically closed of characteristic zero. Consider a family  $\pi : \mathcal{X} \rightarrow B$  of smooth, projective varieties with a reference point  $o \in B$  and a Hodge class  $\gamma \in H^{1,1}(\mathcal{X}_o, \mathbb{Z})$ , where  $\mathcal{X}_b := \pi^{-1}(b)$ . The Hodge locus  $\text{NL}(\gamma) \subset B$  corresponding to  $\gamma$  is the space of all  $b \in B$  such that  $\gamma$  deforms to a Hodge class on  $\mathcal{X}_b$ . By Lefschetz (1,1)-theorem,  $\gamma = c_1(\mathcal{L})$  for some invertible sheaf  $\mathcal{L}$  on  $\mathcal{X}_o$ . We compare  $\text{NL}(\gamma)$  with a Brill-Noether type locus  $\mathcal{B}_{\mathcal{L}}$  associated to  $\mathcal{L}$ . More precisely, when  $h^0(\mathcal{L}) > 1$ , we define  $\mathcal{B}_{\mathcal{L}}$  to be the sub-locus of  $\text{NL}(\gamma)$  consisting of all points  $b \in \text{NL}(\gamma)$  for which  $\mathcal{L}$  deforms to an invertible sheaf  $\mathcal{L}_b$  on  $\mathcal{X}_b$  satisfying  $h^0(\mathcal{L}_b) \geq h^0(\mathcal{L})$ . In some sense,  $\mathcal{B}_{\mathcal{L}}$  consists of those points of  $\text{NL}(\gamma)$  for which the entire linear system  $|\mathcal{L}|$  deforms. The study of these two loci is related to the following classical question:

Given a family  $\pi$  as above and a closed fiber  $\mathcal{X}_o$ , classify effective divisors  $D \subset \mathcal{X}_o$  satisfying the property: for any infinitesimal deformation  $\mathcal{X}_t$  of  $\mathcal{X}_o$  corresponding to  $t \in T_o B$ , the Hodge class  $[D]$  corresponding to  $D$  lifts to a Hodge class on  $\mathcal{X}_t$  if and only if  $D$  lifts to an effective Cartier divisor on  $\mathcal{X}_t$ ?

This question is still wide open. Bloch proved in [Blo72] that semi-regular Cartier divisors satisfy this property. But semi-regularity is a very strong condition and there are several examples of Cartier divisors that are not semi-regular but satisfy this property. In this article, we address the question in terms of the Brill-Noether locus associated to  $\mathcal{L} = \mathcal{O}_{\mathcal{X}_o}(D)$ . In particular, we prove that if the Hodge locus corresponding to  $[D]$  coincides with the Brill-Noether locus  $\mathcal{B}_{\mathcal{L}}$  for  $\mathcal{L} = \mathcal{O}_{\mathcal{X}_o}(D)$ , then  $D$  satisfies the property in the question (see Theorem 3.9). Although we do not prove, but one can observe from the text that in most cases, this condition will in fact exhaustively classify all such divisors.

The other motivation is to study deformation of linear systems. This  $\mathcal{B}_{\mathcal{L}}$  is the correct object to consider for this purpose. One could naively define,  $\mathcal{B}_{\mathcal{L}}$  to be the locus of points  $b \in B$  such that every element of the linear system  $|\mathcal{L}|$  deforms to an effective divisor on

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$\mathcal{X}_b$ . But this will give us the wrong infinitesimal information, meaning the infinitesimal definition of  $\mathcal{B}_{\mathcal{L}}$  will not agree with its global definition. This can be explained using relative Hilbert schemes. More precisely, one expects  $T_o\mathcal{B}_{\mathcal{L}}$  to consist of those tangent vectors  $t \in T_o\mathrm{NL}(\gamma)$  for which every effective divisor of  $|\mathcal{L}|$  lifts to an effective Cartier divisor on  $\mathcal{X}_t$ , where  $\mathcal{X}_t$  is the infinitesimal deformation of  $\mathcal{X}_o$  corresponding to  $t$ . But, quite often, this is not going to be the actual tangent space at  $o$  to  $\mathcal{B}_{\mathcal{L}}$ . In most cases with  $h^0(\mathcal{L}) > 1$ , for any  $b \in \mathrm{NL}(\gamma)$  there exists a deformation of  $\mathcal{L}$  to an invertible sheaf  $\mathcal{L}_b$  on  $\mathcal{X}_b$  satisfying  $h^0(\mathcal{L}_b) > 1$ . In such cases, it is not hard to show that the naive definition of  $\mathcal{B}_{\mathcal{L}}$  will be equal to  $\mathrm{NL}(\gamma)$ . It is possible that the dimension of the linear system  $|\mathcal{L}|$  jumps, i.e., there is an open neighborhood  $U \subset \mathrm{NL}(\gamma)$  of  $o$  such that for all  $u \in U \setminus \{o\}$  and deformation  $\mathcal{L}_u$  of  $\mathcal{L}$  to an invertible sheaf  $\mathcal{L}_u$  on  $\mathcal{X}_u$ ,

$$h^0(\mathcal{L}) > h^0(\mathcal{L}_u).$$

But in this case one observes that  $T_o\mathcal{B}_{\mathcal{L}} \subsetneq T_o\mathrm{NL}(\gamma)$  even when  $\mathrm{NL}(\gamma)$  is smooth at  $o$ . This would mean the dimension of the naive definition of  $\mathcal{B}_{\mathcal{L}}$  is strictly greater than  $\dim T_o\mathcal{B}_{\mathcal{L}}$ , which is not possible (see example in Section 4). To resolve such ambiguity we use the Brill-Noether type definition of  $\mathcal{B}_{\mathcal{L}}$ .

We now discuss the approach taken in this article. For any  $D \in |\mathcal{L}|$ , one can define a class  $\{D\} \in H^0(\mathcal{H}_D^1(\Omega_{\mathcal{X}_o}^1))$ ; this is a classical construction. In fact,  $c_1(\mathcal{L})$  is the image of  $\{D\}$  under the natural homomorphism from  $H^0(\mathcal{H}_D^1(\Omega_{\mathcal{X}_o}^1)) \cong H_D^1(\Omega_{\mathcal{X}_o}^1)$  to  $H^1(\Omega_{\mathcal{X}_o}^1)$ . The tangent space  $T_o\mathrm{NL}(\gamma)$  is given using the cup-product map

$$\cup c_1(\mathcal{L}) : H^1(\mathcal{T}_{\mathcal{X}_o}) \longrightarrow H^2(\mathcal{O}_{\mathcal{X}_o}).$$

In particular, one uses the Kodaira-Spencer map  $\rho_\pi : T_oB \longrightarrow H^1(\mathcal{T}_{\mathcal{X}_o})$  associated to  $\pi$ . Then  $t \in T_o\mathrm{NL}(\gamma)$  if and only if  $\rho_\pi(t) \cup c_1(\mathcal{L}) \longmapsto 0$ . Analogous to the cup-product map, one can define an inner multiplication

$$\lrcorner \{D\} : H^1(\mathcal{T}_{\mathcal{X}_o}) \longrightarrow H_D^2(\mathcal{O}_{\mathcal{X}_o}).$$

We prove that  $t \in T_o\mathcal{B}_{\mathcal{L}}$  if and only if  $\rho_\pi(t) \lrcorner \{D\} = 0$  (Proposition 3.5). As  $\mathcal{L}$  deforms along  $\mathrm{NL}(\gamma)$ , it is possible that the dimension of the space of its global sections drops, sometimes to zero. In such cases  $\mathcal{B}_{\mathcal{L}}$  and  $\mathrm{NL}(\gamma)$  differ, and so do their tangent spaces. Using Lefschetz (1,1)-theorem and deformation theory, there exists an invertible sheaf  $\tilde{\mathcal{L}}$  on  $\pi^{-1}(\mathrm{NL}(\gamma))$  satisfying  $\tilde{\mathcal{L}}|_{\mathcal{X}_o} \cong \mathcal{L}$ . We first prove:

**Theorem 1.1** (Theorem 3.9). *For  $\gamma = c_1(\mathcal{L}) \in H^{1,1}(\mathcal{X}_o, \mathbb{Z})$ , if  $h^0(\tilde{\mathcal{L}}|_{\mathcal{X}_b}) = h^0(\tilde{\mathcal{L}}|_{\mathcal{X}_o})$  for all  $b \in B$ , then  $\mathrm{NL}(\gamma) = \mathcal{B}_{\mathcal{L}}$  and  $T_o\mathrm{NL}(\gamma) = T_o\mathcal{B}_{\mathcal{L}}$ .*

We then produce an example of a family  $\pi$  as above and an invertible sheaf  $\mathcal{L}$  on  $\mathcal{X}_o$  for which the hypothesis of Theorem 1.1 fails. For this example,  $\mathcal{B}_{\mathcal{L}}$  is *properly* contained in  $\mathrm{NL}(\gamma)$  and so is their respective tangent spaces (see Theorem 4.6 and Theorem 4.9). The point to note is that the failure of the hypothesis of Theorem 1.1 does not a-priori imply that  $T_o\mathcal{B}_{\mathcal{L}} \neq T_o\mathrm{NL}(\gamma)$ . It is possible for all *first order* infinitesimal deformation  $\mathcal{X}_t$  of  $\mathcal{X}_o$  corresponding to  $t \in T_o\mathrm{NL}(\gamma)$ , one has  $h^0(\tilde{\mathcal{L}}|_{\mathcal{X}_t}) \geq h^0(\mathcal{L})$ . Of course, there exists higher order infinitesimal deformations  $\mathcal{X}_{t_n}$  of  $\mathcal{X}_o$ , along  $\mathrm{NL}(\gamma)$  with  $h^0(\tilde{\mathcal{L}}|_{\mathcal{X}_{t_n}}) < h^0(\mathcal{L})$ .

We finally produce a family  $\pi : \mathcal{X} \longrightarrow B$  such that  $T_o\mathcal{B}_{\mathcal{L}} \neq T_o\mathrm{NL}(\gamma)$ . To produce such a family we start with a smooth, projective variety  $X$  and an invertible sheaf  $\mathcal{L}$  on  $X$  such that the set of base points  $B$  of  $\mathcal{L}$  is zero dimensional. Choose a point  $p \in B$ ,

and define  $B_p := B \setminus \{p\}$ . We produce a flat family  $\pi : \mathcal{X} \rightarrow X \setminus B_p$  such that for all  $q \in X \setminus B_p$ , the fiber  $\pi^{-1}(q)$  is the blow up of  $X$  at  $B_p \cup q$ . Denote by  $E_q$  the exceptional divisor. There exists an invertible sheaf  $\mathcal{M}$  on  $\mathcal{X}$  such that  $\mathcal{M}_q := \mathcal{M}|_{\mathcal{X}_q} = \mathcal{L}(-E_q)$  for all  $q \in X \setminus B_p$ . We prove that

**Theorem 1.2** (Theorem 4.6 and Theorem 4.9). *Let  $\gamma = c_1(\mathcal{M}_p)$ . Then,  $\dim \mathcal{B}_{\mathcal{M}_p} < \dim \mathrm{NL}(\gamma)$  and  $T_p \mathrm{NL}(\gamma) \neq T_p \mathcal{B}_{\mathcal{M}_p}$ .*

Finally, given a smooth projective surface  $X$ , an invertible sheaf  $\mathcal{L}$  and a positive integer  $n$ , we study the locus of points in  $X$  such that there exists a family of smooth projective curves in the linear system  $|\mathcal{L}|$  of dimension at least  $n$ , passing through these points. We prove that the locus of such points is a Brill-Noether type locus (see Theorem 5.2).

## 2. PRELIMINARIES

We recall some basics on local cohomology groups.

Let  $X$  be a topological space,  $Y \subset X$  a closed subspace and  $\mathcal{F}$  a sheaf of abelian groups on  $X$ . Let  $\Gamma_Y(X, \mathcal{F})$  denote the *group of sections of  $\mathcal{F}$  with support on  $Y$* ; it is also the subgroup of  $\Gamma(X, \mathcal{F})$  consisting of all sections whose support is contained in  $Y$ . Now,  $\Gamma_Y(X, -)$  is a left exact functor from the category of abelian sheaves on  $X$  to abelian groups. We denote the right derived functor of  $\Gamma_Y(X, -)$  by  $H_Y^i(X, -)$ . They are the *cohomology groups of  $X$  with support in  $Y$*  and coefficients in a given sheaf.

For  $\mathcal{F}$  as above, let  $\underline{\Gamma}_Y(\mathcal{F})$  be the sheaf which associates to an open subset  $U$  the abelian group  $\Gamma_{Y \cap U}(U, \mathcal{F}|_U)$ . Denote by  $\mathcal{H}_Y^i(\mathcal{F})$  the associated right derived functor.

Using [HG67, Proposition 1.2] one notices that  $\mathcal{H}_Y^i(\mathcal{F})$  is in fact the sheaf associated to the presheaf which associates the abelian group  $H_{Y \cap U}^i(U, \mathcal{F}|_U)$  to an open subset  $U \subset X$ .

**Lemma 2.1** ([HG67, Corollary 1.1.9]). *Let  $\mathcal{F}$  be a quasi-coherent sheaf on  $X$ . Let  $U := X - Y$  be the complement with  $j : U \hookrightarrow X$  the inclusion. There is a long exact sequence*

$$\begin{aligned} 0 \longrightarrow H_Y^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(U, \mathcal{F}|_U) \longrightarrow H_Y^1(X, \mathcal{F}) \\ \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(U, \mathcal{F}|_U) \longrightarrow H_Y^2(X, \mathcal{F}) \longrightarrow \cdots \end{aligned}$$

*Similarly, there is a short exact sequence*

$$0 \longrightarrow \mathcal{H}_Y^0(X, \mathcal{F}) \longrightarrow \mathcal{H}^0(X, \mathcal{F}) \longrightarrow \mathcal{H}^0(U, \mathcal{F}|_U) \xrightarrow{\delta} \mathcal{H}_Y^1(X, \mathcal{F}) \longrightarrow 0$$

*and  $\mathcal{H}_Y^{i+1}(\mathcal{F}) \cong R^i j_* (\mathcal{F}|_U)$  for all  $i > 0$ .*

**Proposition 2.2.** *Let  $X$  be a scheme,  $Z$  a local complete intersection subscheme in  $X$  and  $\mathcal{F}$  a sheaf of abelian groups on  $X$ . Then the spectral sequence with terms  $E_2^{p,q} = H^p(X, \mathcal{H}_Z^q(X, \mathcal{F}))$  converges to  $H_Z^{p+q}(X, \mathcal{F})$ . Furthermore, if  $\mathcal{F}$  is a locally free  $\mathcal{O}_X$ -module, then  $H_Z^{p+q}(X, \mathcal{F}) \cong H^p(X, \mathcal{H}_Z^q(X, \mathcal{F}))$ , where  $q$  is the codimension of  $Z$  in  $X$  and  $p \geq 0$ .*

*Proof.* The first statement is proven in [HG67, Proposition 1.4].

Assume that  $\mathcal{F}$  is locally free. We will show that  $\mathcal{H}_Z^k(X, \mathcal{F}) = 0$  for  $k \neq q$ .

Since  $Z$  is a local complete intersection subscheme in  $X$ , there exists an affine open covering  $\{U_i\}$  of  $X$  such that for each  $i$  satisfying  $Z \cap U_i \neq \emptyset$ , the  $\mathcal{O}_X(U_i)$ -module  $\mathcal{I}_Z(U_i)$  is generated by a  $\mathcal{O}_X(U_i)$ -regular sequence of length  $q$ . In the terminology of [Har77, Ex.

III.3.4], this is equivalent to the assertion that  $\text{depth}_{\mathcal{I}_Z(U_i)}(\mathcal{O}_X(U_i)) = q$ . By taking a refinement of the covering  $\{U_i\}$  if necessary, we can also assume that

$$\mathcal{F}|_{U_i} \cong \bigoplus_{i=1}^{\text{rk}(\mathcal{F})} \mathcal{O}_{U_i}.$$

This means that for all  $i$  satisfying  $Z \cap U_i \neq \emptyset$ , we have  $\text{depth}_{\mathcal{I}_Z(U_i)}(\mathcal{F}(U_i)) = q$ . Using [Har77, Ex. III.3.3 and 3.4], it follows that  $H_{Z \cap U_i}^k(U_i, \mathcal{F}|_{U_i}) = 0$  for all  $k < q$ .

Now,  $H^k(U_i \setminus Z, \mathcal{F}) \cong H_{Z \cap U_i}^{k+1}(U_i, \mathcal{F}|_{U_i})$  for all  $k \geq 1$  (see [HG67, Proposition 2.2]). By construction, we have  $\mathcal{I}_Z(U_i) = (f_1^{(i)}, \dots, f_q^{(i)})$  if  $Z \cap U_i \neq \emptyset$ . Hence, any such complement  $U_i \setminus Z$  can be covered by  $q$  open affine sets,  $V_j^{(i)} := D(f_j^{(i)})$  for  $j = 1, \dots, q$ . Then, [Har77, III. Ex. 4.8] implies that  $H^k(U_i \setminus Z, \mathcal{F}) = 0$  for  $k \geq q$ , and hence  $H_{Z \cap U_i}^k(U_i, \mathcal{F}|_{U_i}) = 0$  for all  $k \geq q + 1$ .

As  $\mathcal{H}_Z^k(X, \mathcal{F})$  is supported on  $Z$ , this means that  $\mathcal{H}_Z^k(X, \mathcal{F}) = 0$  for  $k \neq q$ . Since

$$E_2^{p,q} = H^p(X, \mathcal{H}_Z^q(X, \mathcal{F})) \Rightarrow H_Z^{p+q}(X, \mathcal{F}),$$

we conclude that  $H_Z^{p+q}(X, \mathcal{F}) \cong H^p(X, \mathcal{H}_Z^q(\mathcal{F}))$ . This completes the proof.  $\square$

### 3. RELATIVE BRILL-NOETHER TYPE LOCUS

Let  $X$  be a smooth projective surface in  $\mathbb{P}^3$ , and let  $\mathcal{L}$  be an invertible sheaf on  $X$  with  $h^0(\mathcal{L}) > 1$ . Assume that there is a reduced divisor on  $X$  lying in the complete linear system  $|\mathcal{L}|$ . Let  $\{U_i\}_{i \in I}$  be a Zariski open affine covering of  $X$  such that  $D \cap U_i$  is defined by a single equation, say  $f_i = 0$  with  $f_i \in \Gamma(U_i, \mathcal{O}_X)$ . Denote by  $V_i$  the open affine set  $U_i \setminus \{f_i = 0\}$ .

To describe the cohomology class of  $D$  in  $H^2(X, \mathbb{Z})$ , using Lemma 2.1 we have the exact sequences

$$\dots \longrightarrow \Gamma(U_i, \Omega_X^1) \xrightarrow{\delta'_i} \Gamma(V_i, \Omega_X^1) \xrightarrow{\delta_i} \Gamma(U_i, \mathcal{H}_D^1(\Omega_X^1)) \longrightarrow \dots \quad (3.1)$$

Notice that the sections  $\delta_i(df_i/f_i) \in \Gamma(U_i, \mathcal{H}_D^1(\Omega_X^1))$  agree on the intersections  $U_{ij} := U_i \cap U_j$ , i.e.,

$$\delta_i(df_i/f_i)|_{U_{ij}} = \delta_j(df_j/f_j)|_{U_{ij}}.$$

Indeed,  $f_i|_{U_{ij}} = \lambda_{ij}f_j|_{U_{ij}}$  for some  $\lambda_{ij} \in \Gamma(U_{ij}, \mathcal{O}_{U_{ij}}^\times)$ . Then,

$$\frac{df_i}{f_i} \Big|_{U_{ij}} = \frac{d\lambda_{ij}}{\lambda_{ij}} \Big|_{U_{ij}} + \frac{df_j}{f_j} \Big|_{U_{ij}}.$$

As  $\lambda_{ij}$  is invertible,  $d\lambda_{ij}/\lambda_{ij} \in \Gamma(U_{ij}, \Omega_X^1)$ . Using the short exact sequence (3.1) this implies that  $\delta_j \circ \delta'_j|_{U_{ij}}(d\lambda_{ij}/\lambda_{ij}) = 0$ . Hence,  $\delta_i(df_i/f_i)|_{U_{ij}} = \delta_j(df_j/f_j)|_{U_{ij}}$ .

Therefore, the local sections  $\delta_i(df_i/f_i) \in \Gamma(U_i, \mathcal{H}_D^1(\mathcal{O}_X))$  glue compatibly to define a global section  $\{D\} \in H^0(X, \mathcal{H}_D^1(\Omega_X^1))$ .

Using the short exact sequence in (3.1) and arguing as above, it is easy to see that the above class  $\{D\}$  does not depend on the choice of the representatives  $f_i$ .

**Remark 3.1.** Proposition 2.2 implies that  $H^0(\mathcal{H}_D^1(\Omega_X^1)) \cong H_D^1(\Omega_X^1)$ , while Lemma 2.1 implies that we have a homomorphism  $H_D^1(\Omega_X^1) \rightarrow H^1(\Omega_X^1)$ . The Chern class

$$c_1(\mathcal{L}) \in H^1(X, \Omega_X^1)$$

is the image of  $\{D\} \in H_D^1(\Omega_X^1)$  under this homomorphism (see [FGA62]).

We now describe the cup-product  $\cup_{c_1(\mathcal{L})} : H^1(\mathcal{T}_X) \rightarrow H^2(\mathcal{O}_X)$ . Define  $U := X \setminus D$ , and let  $j : U \hookrightarrow X$  be the open immersion. The proof of Proposition 2.2 shows that  $\mathcal{H}_D^0(\mathcal{O}_X) = 0$ . By Lemma 2.1, we have a short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow j_* \mathcal{O}_U \xrightarrow{\delta} \mathcal{H}_D^1(\mathcal{O}_X) \rightarrow 0. \quad (3.2)$$

Let

$$\lrcorner \{D\} : \mathcal{T}_X \rightarrow \mathcal{H}_D^1(\mathcal{O}_X)$$

be the homomorphism which on each  $U_i$  is defined as

$$\phi \mapsto \delta \left( \frac{\phi(df_i)}{f_i} \right),$$

where  $\delta$  is the projection in (3.2). Proposition 2.2 implies that  $H^1(\mathcal{H}_D^1(\mathcal{O}_X)) \cong H_D^2(\mathcal{O}_X)$ . This induces a homomorphism

$$\lrcorner \{D\} : H^1(\mathcal{T}_X) \rightarrow H^1(X, \mathcal{H}_D^1(\mathcal{O}_X)) \cong H_D^2(\mathcal{O}_X);$$

with a slight abuse of notation, this will also be denoted by  $\lrcorner \{D\}$ . Using Remark 3.1 one can check that the cup-product map  $\cup_{c_1(\mathcal{L})}$  is the composition

$$\cup_{c_1(\mathcal{L})} : H^1(\mathcal{T}_X) \xrightarrow{\lrcorner \{D\}} H_D^2(\mathcal{O}_X) \rightarrow H^2(\mathcal{O}_X). \quad (3.3)$$

Let

$$\lrcorner \{D\}' : \mathcal{N}_{D|X} \rightarrow \mathcal{H}_D^1(\mathcal{O}_X)$$

be the homomorphism which on each  $U_i$  is defined as

$$\lrcorner \{D\}'|_{U_i}(\phi) = \delta \left( \frac{\widetilde{\phi(f_i)}}{f_i} \right),$$

where  $\widetilde{g}$  for any  $g \in \mathcal{O}_D(U_i \cap D)$  is its preimage under the natural surjective homomorphism  $\mathcal{O}_X(U_i) \rightarrow \mathcal{O}_D(U_i \cap D)$ . It follows from the short exact sequence in (3.2) that the map  $\lrcorner \{D\}'$  does not depend on the choice of the lift of  $\phi(f_i)$ .

**Lemma 3.2.** *The above homomorphism  $\lrcorner \{D\}' : \mathcal{N}_{D|X} \rightarrow \mathcal{H}_D^1(\mathcal{O}_X)$  is injective.*

*Proof.* From the short exact sequence in (3.2) it follows that  $\delta \left( \frac{\widetilde{\phi(f_i)}}{f_i} \right) = 0$  on  $U_i$  if and only if  $\widetilde{\phi(f_i)}/f_i \in \mathcal{O}_X(U_i)$ , which is possible if and only if  $\widetilde{\phi(f_i)} \in \mathcal{O}_X(-D)(U_i)$ . But this means that  $\phi(f_i) = 0$ ; hence  $\phi = 0$  because it is determined by its evaluation on  $f_i$ . Consequently,  $\lrcorner \{D\}'$  is injective.  $\square$

**Corollary 3.3.** *There is a short exact sequence*

$$0 \rightarrow \mathcal{N}_{D|X} \xrightarrow{\lrcorner \{D\}'} \mathcal{H}_D^1(\mathcal{O}_X) \xrightarrow{\psi} \mathcal{H}_D^1(\mathcal{O}_X(D)) \rightarrow 0, \quad (3.4)$$

where  $\psi$  is the homomorphism arising from the natural homomorphism  $\mathcal{O}_X \rightarrow \mathcal{O}_X(D)$ .

*Proof.* The injectivity of  $\lrcorner\{D\}'$  is proved in Lemma 3.2. Clearly,  $j_*\mathcal{O}_U \cong j_*(\mathcal{O}_X(D)|_U)$ . So, we have the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & j_*\mathcal{O}_U & \xrightarrow{\delta} & \mathcal{H}_D^1(\mathcal{O}_X) \longrightarrow 0 \\
& & \downarrow & & \downarrow \text{id} & & \downarrow \psi \\
0 & \longrightarrow & \mathcal{O}_X(D) & \longrightarrow & j_*\mathcal{O}_U & \xrightarrow{\delta'} & \mathcal{H}_D^1(\mathcal{O}_X(D)) \longrightarrow 0
\end{array}$$

$\circlearrowleft \quad \quad \quad \circlearrowleft$

where the horizontal short exact sequences are obtained using Lemma 2.1 and the above identification  $j_*\mathcal{O}_U \cong j_*(\mathcal{O}_X(D)|_U)$ . Applying Snake lemma to the above diagram, the homomorphism  $\psi$  is surjective and  $\ker(\psi)$  is isomorphic to the cokernel of the homomorphism  $\mathcal{O}_X \rightarrow \mathcal{O}_X(D)$ , which is  $\mathcal{N}_{D|X}$  by the Poincaré adjunction formula. It just remains to prove that the induced homomorphism from  $\mathcal{N}_{D|X}$  to  $\mathcal{H}_D^1(\mathcal{O}_X)$  is  $\lrcorner\{D\}'$  or, equivalently, exactness in the middle of (3.4).

Using the above diagram, we have

$$\ker \psi = \delta(\ker \delta') = \delta(\text{Im}(\mathcal{O}_X(D) \rightarrow j_*\mathcal{O}_U)).$$

Consider the homomorphism  $\mathcal{O}_X(D) \rightarrow \mathcal{N}_{D|X}$  defined on open subsets  $U_i$  by  $g_i/f_i \mapsto \phi$ , where  $g_i \in \mathcal{O}_X(U_i)$  while  $\phi$  is defined as  $f_i \mapsto g_i \bmod \mathcal{I}_D(U_i)$ . For such  $\phi$ , the definition of  $\lrcorner\{D\}'$  states that  $\lrcorner\{D\}'|_{U_i}(\phi) = \delta(g_i/f_i)$ . So, for each  $g_i/f_i \in \mathcal{O}_X(D)$  we can construct  $\phi$  as above such that  $\lrcorner\{D\}'|_{U_i}(\phi) = \delta(g_i/f_i)$ . Observe that the induced map from  $\mathcal{O}_X(D)$  to  $\mathcal{N}_{D|X}$  is surjective. Hence,  $\ker \psi = \text{Im} \lrcorner\{D\}'$ . This completes proof.  $\square$

The following lemma will be used in the proof of Proposition 3.5 below.

**Lemma 3.4.** *Let  $i : D \rightarrow X$  be the closed immersion. Then*

$$\mathcal{E}xt_X^m(\mathcal{H}_D^1(\mathcal{O}_X(D)), i_*\mathcal{N}_{D|X}) = 0 \quad \text{for } m = 0, 1.$$

*In particular,  $\text{Ext}_X^1(\mathcal{H}_D^1(\mathcal{O}_X(D)), i_*\mathcal{N}_{D|X}) = 0$ .*

*Proof.* By adjunction,

$$\mathcal{E}xt_X^m(\mathcal{H}_D^1(\mathcal{O}_X(D)), i_*\mathcal{N}_{D|X}) \cong \mathcal{E}xt_D^m(\mathcal{H}_D^1(\mathcal{O}_X(D)) \otimes_{\mathcal{O}_X} \mathcal{O}_D, \mathcal{N}_{D|X}).$$

We claim that  $\mathcal{H}_D^1(\mathcal{O}_X(D)) \otimes_{\mathcal{O}_X} \mathcal{O}_D = 0$ .

To prove the claim, using Lemma 2.1 we have the short exact sequence

$$0 \rightarrow \mathcal{O}_X(D) \rightarrow j_*(\mathcal{O}_X(D)|_U) \xrightarrow{\delta} \mathcal{H}_D^1(\mathcal{O}_X(D)) \rightarrow 0.$$

So,  $\mathcal{H}_D^1(\mathcal{O}_X(D))$  is supported on  $D$ . For any  $x \in D$ ,

$$(j_*\mathcal{O}_X(D)|_U)_x \cong \mathcal{O}_{X,x}[1/f_x],$$

where  $f_x \in \mathcal{O}_{X,x}$  is the defining equation for  $D$  at  $x$ . Any element of  $\mathcal{H}_D^1(\mathcal{O}_X(D))_x$  is of the form  $\delta(g) = f_x \delta(g/f_x)$ , where  $g \in j_*(\mathcal{O}_X(D)|_U)_x$ . So,  $\delta(g) \otimes_{\mathcal{O}_{X,x}} 1$  is zero in  $\mathcal{H}_D^1(\mathcal{O}_X)_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/(f_x)$ , which implies that  $\mathcal{H}_D^1(\mathcal{O}_X)_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathcal{I}_{D,x} = 0$ . This proves the claim.

Hence,  $\mathcal{E}xt_D^m(\mathcal{H}_D^1(\mathcal{O}_X(D)) \otimes_{\mathcal{O}_X} \mathcal{O}_D, \mathcal{N}_{D|X}) = 0$  for  $m = 0, 1$ . Hence,

$$\mathcal{E}xt_X^m(\mathcal{H}_D^1(\mathcal{O}_X(D)), i_*\mathcal{N}_{D|X}) = 0 \quad \text{for } m = 0, 1.$$

By Grothendieck Spectral sequence,

$$\mathrm{Ext}_X^1(\mathcal{H}_D^1(\mathcal{O}_X(D)), i_* \mathcal{N}_{D|X}) \cong \bigoplus_{i=0}^1 H^i(\mathcal{E}xt_X^{1-i}(\mathcal{H}_D^1(\mathcal{O}_X(D)), i_* \mathcal{N}_{D|X})).$$

Hence,  $\mathrm{Ext}_X^1(\mathcal{H}_D^1(\mathcal{O}_X(D)), i_* \mathcal{N}_{D|X}) = 0$ , proving the lemma.  $\square$

Given any  $t \in H^1(\mathcal{T}_X)$ , denote by  $X_t$  the infinitesimal deformation of  $X$  along  $t$ .

**Proposition 3.5.** *For any reduced  $D \in |\mathcal{L}|$ , the homomorphism  $\cup_{C_1}(\mathcal{L})$  factors through  $\lrcorner\{D\}$ , meaning the following diagram is commutative:*

$$\begin{array}{ccc} H^1(\mathcal{T}_X) & & \\ \cup_{C_1}(\mathcal{L}) \downarrow & \searrow \lrcorner\{D\} & \\ H^2(\mathcal{O}_X) & \longleftarrow & H_D^2(\mathcal{O}_X) \end{array} \quad (3.5)$$

Furthermore, given any  $t \in \ker \cup_{C_1}(\mathcal{L})$  and  $X_t$  the corresponding infinitesimal deformation of  $X$ , the divisor  $D$  lifts to an effective Cartier divisor in  $X_t$  if and only if  $t \lrcorner\{D\} = 0$ .

*Proof.* By [Blo72, Proposition 6.2] there is a commutative diagram

$$\begin{array}{ccc} H^1(\mathcal{T}_X) & & \\ u^* \downarrow & \searrow \lrcorner\{D\} & \\ H^1(\mathcal{N}_{D|X}) & \xrightarrow{\lrcorner\{D\}'} & H_D^2(\mathcal{O}_X) \end{array}$$

where  $u^*$  is the composition  $H^1(\mathcal{T}_X) \rightarrow H^1(\mathcal{T}_X \otimes \mathcal{O}_D) \rightarrow H^1(\mathcal{N}_{D|X})$ ; the first homomorphism in this composition is induced by restriction and the second one by the natural homomorphism  $\mathcal{T}_X \otimes \mathcal{O}_D \rightarrow \mathcal{N}_{D|X}$ . The commutativity of (3.5) then follows from the definition of cup-product map given in (3.3). This proves the first part of the proposition.

Using [Blo72, Proposition 2.6], the divisor  $D$  lifts to an effective divisor in  $X_t$  if and only if  $u^*(t) = 0$ ; as before,  $X_t$  the infinitesimal deformation of  $X$  along  $t$ . From Lemma 3.4 it follows that

$$\mathrm{Ext}_X^1(\mathcal{H}_D^1(\mathcal{O}_X(D)), i_* \mathcal{N}_{D|X}) = 0.$$

This implies that the short exact sequence in (3.4) splits. Hence, the induced homomorphism of global sections  $H^0(\mathcal{H}_D^1(\mathcal{O}_X)) \rightarrow H^0(\mathcal{H}_D^1(\mathcal{O}_X(D)))$  is surjective. This means that

$$\lrcorner\{D\}' : H^1(\mathcal{N}_{D|X}) \rightarrow H^1(\mathcal{H}_D^1(\mathcal{O}_X)) \cong H_D^2(\mathcal{O}_X)$$

is injective. Hence,  $D$  lifts to an effective divisor in  $X_t$  if and only if  $\lrcorner\{D\}(t) = 0$ . This completes the proof.  $\square$

We now apply Proposition 3.5 to families of smooth projective varieties.

Let  $\pi : \mathcal{X} \rightarrow B$  be a flat family of smooth projective varieties with  $X$  being the fiber over a base point  $o \in B$ . For any  $u \in B$ , denote the fiber  $\pi^{-1}(u)$  by  $X_u$ .

**Remark 3.6.** The differential of  $\pi$  produces a short exact sequence

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_X|_X = \mathcal{T}_X \otimes \mathcal{O}_X \longrightarrow \pi^*T_oB \longrightarrow 0,$$

where  $T_oB$  is the tangent space to  $B$  at  $o$ . The *Kodaira-Spencer map*

$$\rho_\pi : T_oB \cong H^0(\pi^*(T_oB)) \longrightarrow H^1(\mathcal{T}_X)$$

is the boundary homomorphism associated to the above short exact sequence. For an algebraic line bundle  $\mathcal{L}$  on  $X$ , denote by  $\text{NL}(\gamma)$  the *Hodge locus* corresponding to the Hodge class

$$\gamma := c_1(\mathcal{L}) \in H^{1,1}(X, \mathbb{Q})$$

(see [Voi03] for definition of Hodge locus). Define  $\mathcal{X}' := \pi^{-1}(\text{NL}(\gamma))$ . After contracting  $B$  if necessary, there is an invertible sheaf  $\tilde{\mathcal{L}}$  on  $\mathcal{X}'$  such that  $\tilde{\mathcal{L}}|_X \cong \mathcal{L}$  (see [Ser06, § 3.3.1]).

For any  $t \in T_oB$ , let  $X_t \subset \mathcal{X}$  the infinitesimal deformation of  $X$  along  $t$ . For any  $u \in \text{NL}(\gamma)$ , define  $\tilde{\mathcal{L}}_u := \tilde{\mathcal{L}}|_{X_u}$ .

Given any  $b \in B$  and an invertible sheaf  $\mathcal{L}_b$  on  $X_b$ , we say that  $\mathcal{L}$  *deforms to an invertible sheaf  $\mathcal{L}_b$  on  $X_b$*  if there exists a connected closed subscheme  $W \subset B$  containing both  $o$  and  $b$ , and an invertible sheaf  $\mathcal{L}_W$  on  $X_W$ , such that  $\mathcal{L}_W|_X \cong \mathcal{L}$  and  $\mathcal{L}_W|_{X_b} \cong \mathcal{L}_b$ .

From the upper-semicontinuity theorem for the dimension of global sections it follows that  $\mathcal{B}_{\mathcal{L}}$  is a closed subscheme in  $\text{NL}(\gamma)$ .

Given a family  $\pi$  and  $\gamma$  as before, the *Brill-Noether sub-locus of  $\text{NL}(\gamma)$  associated to  $\mathcal{L}$*  is the subset  $\mathcal{B}_{\mathcal{L}} \subset B$  consisting of all  $b \in B$  such that there exists a connected closed subscheme  $W \subset B$  containing both the points  $o$  and  $b$ , and an invertible sheaf  $\mathcal{L}_W$  on  $X_W$ , such that

- (1)  $\mathcal{L}_W|_X \cong \mathcal{L}$ , and
- (2)  $h^0(\mathcal{L}_W|_{X_w}) \geq h^0(\mathcal{L})$  for all  $w \in W$ .

From Lefschetz (1,1)-theorem it follows that  $\mathcal{B}_{\mathcal{L}} \subset \text{NL}(\gamma)$ .

**Proposition 3.7.** *The tangent space at the point  $o \in \mathcal{B}_{\mathcal{L}}$  is*

$$T_o\mathcal{B}_{\mathcal{L}} = \rho_\pi^{-1} \left( \bigcap_{\substack{D \in |\mathcal{L}| \\ \text{reduced}}} \lrcorner \{D\} \right).$$

*Proof.* For any  $t \in T_oB$ , we have  $t \in T_o\text{NL}(\gamma)$  if and only if  $\mathcal{L}$  lifts to an invertible sheaf  $\mathcal{L}_t$  on  $X_t$ , where  $X_t$  is the infinitesimal deformation of  $X$  along  $t$  [Ser06, § 3.3.1]. By definition,  $t \in T_o\mathcal{B}_{\mathcal{L}}$  if and only if  $t \in T_o\text{NL}(\gamma)$  and

$$\dim_{k[\epsilon]/(\epsilon^2)} H^0(\mathcal{L}_t) \geq h^0(\mathcal{L}). \quad (3.6)$$

By Proposition 3.5, for all reduced divisor  $D \in |\mathcal{L}|$ ,

$$\rho_\pi(t) \lrcorner \{D\} = 0$$

if and only if  $D$  lifts to an effective Cartier divisor on  $X_t$ . Now this is possible if and only if the natural restriction homomorphism  $H^0(\mathcal{L}_t) \longrightarrow H^0(\mathcal{L})$  is surjective. By the long



exact sequence of cohomologies associated to

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_t \xrightarrow{\text{mod } \mathfrak{t}} \mathcal{L} \longrightarrow 0, \quad (3.7)$$

this is equivalent to the statement that

$$\dim_k H^0(\mathcal{L}_t) = 2h^0(\mathcal{L}).$$

Now, we have  $2 \dim_{k[\epsilon]/(\epsilon^2)} H^0(\mathcal{L}_t) = \dim_k H^0(\mathcal{L}_t)$ . Therefore, for  $t \in T_o \text{NL}(\gamma)$ , the inequality in (3.6) holds if and only if  $\rho_\pi(t) \lrcorner \{D\} = 0$  for all reduced  $D \in |\mathcal{L}|$ . This completes the proof.  $\square$

It should be mentioned that it is not true that the tangent space  $T_b \mathcal{B}_{\mathcal{L}}$  can be described by a Kodaira-Spencer type formula  $T_b B \longrightarrow H^1(\mathcal{T}_{X_b})$ ; it is possible that

$$T_b \mathcal{B}_{\mathcal{L}} \neq \rho_\pi^{-1} \left( \bigcap_{\substack{D \in |\mathcal{L}_b| \\ \text{reduced}}} \lrcorner \{D\} \right).$$

However the following is true.

**Corollary 3.8.** *Suppose that  $h^1(\mathcal{O}_{X_b}) = 0$  for all  $b \in \text{NL}(\gamma)$ . Then, for any  $b \in \mathcal{B}_{\mathcal{L}}$ , and a deformation  $\mathcal{L}_b$  of  $\mathcal{L}$  on  $X_b$  with  $h^0(\mathcal{L}_b) = h^0(\mathcal{L})$  such that a general element of  $|\mathcal{L}_b|$  is reduced,*

$$T_b \mathcal{B}_{\mathcal{L}} = \rho_\pi^{-1} \left( \bigcap_{\substack{D \in |\mathcal{L}_b| \\ \text{reduced}}} \lrcorner \{D\} \right),$$

where  $\rho_\pi : T_b B \longrightarrow H^1(\mathcal{T}_{X_b})$  is the associated Kodaira-Spencer map.

*Proof.* Since  $h^1(\mathcal{O}_{X_u}) = 0$  for all  $u \in \text{NL}(\gamma)$ , there is an unique deformation  $\mathcal{L}_b$  of  $\mathcal{L}$  to an invertible sheaf on  $X_b$ . By assumption we have  $h^0(\mathcal{L}_b) = h^0(\mathcal{L})$ . Since the complete linear system  $\mathcal{L}$  on  $\mathcal{X}_o$  deforms to the complete linear system  $H^0(\mathcal{L}_b)$  in this case, observe that  $T_b \mathcal{B}_{\mathcal{L}} = T_b \mathcal{B}_{\mathcal{L}_b}$ . Using the arguments in the proof of Proposition 3.7 it follows that

$$T_b \mathcal{B}_{\mathcal{L}} = T_b \mathcal{B}_{\mathcal{L}_b} = \rho_\pi^{-1} \left( \bigcap_{\substack{D \in |\mathcal{L}_b| \\ \text{reduced}}} \lrcorner \{D\} \right).$$

This completes the proof.  $\square$

**Theorem 3.9.** *If there is an open neighborhood  $U \subset \text{NL}(\gamma)$  of  $o$  such that  $h^0(\tilde{\mathcal{L}}_u) = h^0(\mathcal{L})$  for all  $u \in U$ , then  $\text{NL}(\gamma) \cap U = \mathcal{B}_{\mathcal{L}} \cap U$  and  $T_u \text{NL}(\gamma) = T_u \mathcal{B}_{\mathcal{L}}$  for all  $u \in U$ .*

*Proof.* By the hypothesis on the theorem,  $\text{NL}(\gamma) \cap U = \mathcal{B}_{\mathcal{L}} \cap U$ . To prove the statement on the tangent space, note that

$$\dim_{k[\epsilon]/(\epsilon^2)} H^0(\mathcal{L}_t) = h^0(\mathcal{L})$$

for all  $t \in T_o \text{NL}(\gamma)$ , because  $h^0(\mathcal{L}_u) = h^0(\mathcal{L})$  for all  $u \in U$ . Using the long exact sequence of cohomologies associated to (3.7),

$$\dim_k H^0(\mathcal{L}_t) \leq 2h^0(\mathcal{L})$$

with the equality holding if and only if the induced homomorphism  $H^0(\mathcal{L}_t) \rightarrow H^0(\mathcal{L})$  is surjective. Now,  $\dim_k H^0(\mathcal{L}_t) = 2 \dim_{k[\epsilon]/(\epsilon^2)} H^0(\mathcal{L}_t)$ , and  $\dim_{k[\epsilon]/(\epsilon^2)} H^0(\mathcal{L}_t) \geq r$ . Hence,  $H^0(\mathcal{L}_t) \rightarrow H^0(\mathcal{L})$  is surjective. In other words, every  $D \in |\mathcal{L}|$  lifts to an effective Cartier divisor on  $X_t$ . Then, Proposition 3.5 implies that  $t_\perp\{D\} = 0$  for any  $t \in T_o \text{NL}(\gamma)$  and any reduced  $D \in |\mathcal{L}|$ . Now,  $T_o \text{NL}(\gamma) \subset T_o(\mathcal{B}_{\mathcal{L}})$  by Proposition 3.7. On the other hand, from the diagram (3.5) it follows that the reverse inclusion holds. Hence we conclude that  $T_o \text{NL}(\gamma) = T_o(\mathcal{B}_{\mathcal{L}})$ . Similarly, using the proof of Corollary 3.8, it can be proved that  $T_u \text{NL}(\gamma) = T_u \mathcal{B}_{\mathcal{L}}$  for all  $u \in U$ . This completes the proof.  $\square$

Observe that when  $\dim_o \mathcal{B}_{\mathcal{L}} < \dim_o \text{NL}(\gamma)$ , it is not obvious  $T_o \mathcal{B}_{\mathcal{L}} \subsetneq T_o \text{NL}(\gamma)$ . In particular, there are examples of families of smooth projective families  $\pi : \mathcal{X} \rightarrow B$  and an invertible sheaf  $\mathcal{L}$  on  $\mathcal{X}$  such that there exists a point  $o \in B$  for which

- (1)  $h^0(\mathcal{L}|_{\mathcal{X}_o}) > h^0(\mathcal{L}|_{\mathcal{X}_u})$  for all  $u \in B \setminus \{o\}$ ,
- (2) but  $h^0(\mathcal{L}|_{\mathcal{X}_t}) = h^0(\mathcal{L}|_{\mathcal{X}_o})$  for all *first order* infinitesimal deformation  $\mathcal{X}_t$  of  $\mathcal{X}_o$ ,  $t \in T_o B$ .

Of course, for any such  $\pi$  and invertible sheaf  $\mathcal{L}$ , there exists some higher order infinitesimal deformation  $\mathcal{X}_{t'}$  of  $\mathcal{X}_o$  for which  $h^0(\mathcal{L}|_{\mathcal{X}_{t'}}) < h^0(\mathcal{L}|_{\mathcal{X}_o})$ . However, in the next section, we produce examples of  $\pi$  and  $\mathcal{L}$  such that (1) holds as before and furthermore, there exists  $t \in T_o B$  such that  $h^0(\mathcal{L}|_{\mathcal{X}_t}) < h^0(\mathcal{L}|_{\mathcal{X}_o})$ .

#### 4. JUMPING LOCUS OF LINEAR SYSTEMS

In this section, we produce a family  $\pi : \mathcal{X} \rightarrow B$  and an invertible sheaf  $\mathcal{M}$  on  $\mathcal{X}$  such that there exists a point  $o \in B$  for which

$$B_{\mathcal{M}_o} \subsetneq \text{NL}(c_1(\mathcal{M}_o)) = B$$

and  $T_o B_{\mathcal{M}_o} \subsetneq T_o \text{NL}(\gamma)$ . This gives an example of a classical question: Given a family of smooth, projective varieties  $\pi : \mathcal{X} \rightarrow B$ , when does there exist a closed fiber  $\mathcal{X}_o$  and an effective divisor  $D \subset \mathcal{X}_o$  such that there is an infinitesimal deformation  $\mathcal{X}_t$  of  $\mathcal{X}_o$  along some tangent  $t \in T_o B$  for which the Hodge class  $[D] \in H^{1,1}(\mathcal{X}_o, \mathbb{Q})$  lifts to a Hodge class on  $\mathcal{X}_t$  but  $D$  does not lift to an effective Cartier divisor on  $\mathcal{X}_t$ ?

**Setup 4.1.** Let  $Y$  be a smooth projective variety of dimension at least 2, and let  $\mathcal{L}$  be an invertible sheaf on  $Y$ . Suppose that the base locus  $B$  of  $H^0(\mathcal{L})$  is a (finite) collection of closed points containing a point  $p$  with multiplicity 1. Define  $B_p := B \setminus \{p\}$  and  $Z := Y \setminus B_p$ . Consider the closed subscheme  $B_p \times Y + \Delta \subset Y \times Y$ , where  $\Delta \subset Y \times Y$  is the diagonal, and define

$$E_0 := (B_p \times Y + \Delta) \cap (Y \times Z).$$

**Notation 4.2.** Let  $\bar{\pi} : \bar{Y} \rightarrow Y \times Z$  be the blow-up along  $E_0$ . The exceptional divisor will be denoted by  $E$ . Define

$$p_i := \text{pr}_i \circ \bar{\pi},$$

where  $\text{pr}_1$  (respectively,  $\text{pr}_2$ ) is the projection of  $Y \times Z$  to  $Y$  (respectively,  $Z$ ). Let

$$\mathcal{M} := p_1^* \mathcal{L} \otimes \mathcal{O}_{\bar{Y}}(-E)$$

the invertible sheaf on  $\bar{Y}$ .

**Lemma 4.3.** *The morphism  $p_2 : \bar{Y} \rightarrow Z$  is flat. Furthermore,  $p_2|_E$  is flat.*

*Proof.* This is because  $E_0$  is flat over  $Z$  under the second projection map  $\text{pr}_2$ .  $\square$

**Notation 4.4.** In Remark 3.6, replace the family  $\pi : \mathcal{X} \rightarrow B$  by  $p_2$ . We have the corresponding Kodaira-Spencer map  $\rho_\pi : T_p Y \rightarrow H^1(\mathcal{T}_{\overline{Y}_p})$ .

**Notation 4.5.** Denote by  $\overline{Y}_q$  (respectively,  $E(q)$ ) the fiber over  $q$  under the morphism  $p_2$  (respectively,  $p_2|_E$ ). Define  $\mathcal{M}_q := \mathcal{M} \otimes \overline{Y}_q$  and  $p_1(q) := p_1|_{\overline{Y}_q}$ . Observe that this morphism  $p_1(q)$  is surjective as it is simply the blow-up of  $Y$  along  $B_p \cup q$ .

**Theorem 4.6.** *The inequality  $h^0(\mathcal{M}_y) < h^0(\mathcal{M}_p)$  holds for any  $y \neq p$ . In particular, for  $\gamma = c_1(\mathcal{M}_p)$  on  $\overline{Y}_p$ ,*

$$\mathcal{B}_{\mathcal{M}_p} \neq \text{NL}(\gamma) = Z.$$

*Proof.* Consider the short exact sequence

$$0 \rightarrow \mathcal{I}_{E_0} \xrightarrow{h} \mathcal{O}_{Y \times Z} \rightarrow \mathcal{O}_{E_0} \rightarrow 0.$$

Let

$$\overline{\pi}^* \mathcal{I}_{E_0} \xrightarrow{\overline{\pi}^*(h)} \mathcal{O}_{\overline{Y}} \rightarrow \mathcal{O}_E \rightarrow 0$$

be its pull back by  $\overline{\pi}$ . Note that the image of the homomorphism  $\overline{\pi}^*(h)$  is  $\mathcal{I}_E$ . Hence there is the short exact sequence

$$0 \rightarrow \mathcal{O}_{\overline{Y}}(-E) \rightarrow \mathcal{O}_{\overline{Y}} \rightarrow \mathcal{O}_E \rightarrow 0.$$

Tensoring it by  $p_1^* \mathcal{L}$ ,

$$0 \rightarrow \mathcal{M} \rightarrow p_1^* \mathcal{L} \rightarrow p_1^* \mathcal{L} \otimes \mathcal{O}_E \rightarrow 0. \quad (4.1)$$

Now,

$$p_1^* \mathcal{L} \otimes \mathcal{O}_{\overline{Y}_q} \cong p_1(q)^* \mathcal{L}, \quad p_1^* \mathcal{L} \otimes \mathcal{O}_E \otimes \mathcal{O}_{\overline{Y}_q} \cong p_1(q)^* \mathcal{L} \otimes \mathcal{O}_{E(q)}.$$

By Lemma 4.3, the restriction  $p_2|_E$  is flat. As  $p_1^* \mathcal{L}$  is an invertible sheaf on  $\overline{Y}$  which is flat over  $Z$  via  $p_2$ , it follows that  $(p_1^* \mathcal{L} \otimes \mathcal{O}_E)_y$  is  $\mathcal{O}_{Z,q}$ -flat for any  $y \in \overline{Y}_q$ . Hence,  $\text{Tor}_{\mathcal{O}_{Z,q}}^1((p_1^* \mathcal{L} \otimes \mathcal{O}_E)_y, k(q)) = 0$  for any  $y \in \overline{Y}_q$ . Tensoring (4.1) by  $\mathcal{O}_{\overline{Y}_q}$ , yields the short exact sequence

$$0 \rightarrow \mathcal{M}_q \rightarrow p_1(q)^* \mathcal{L} \rightarrow p_1(q)^* \mathcal{L} \otimes \mathcal{O}_{E(q)} \rightarrow 0. \quad (4.2)$$

As the morphism  $p_1(q)$  is surjective,

$$H^0(p_1(q)^* \mathcal{L} \otimes \mathcal{O}_{E(q)}) = H^0(p_1(q)_*(p_1(q)^* \mathcal{L} \otimes \mathcal{O}_{E(q)}))$$

and

$$H^0(p_1(q)^* \mathcal{L}) = H^0(p_1(q)_*(p_1(q)^* \mathcal{L})).$$

Define  $E_0(q) := E_0 \cap Y \times q$ . As  $p_1(q)$  is the blow-up map of  $Y$  along  $B_p \cup q$ , it follows that  $p_1(q)^* \mathcal{O}_{E_0(q)} \cong \mathcal{O}_{E(q)}$ . Using the projection formula,

$$H^0(p_1(q)^* \mathcal{L} \otimes \mathcal{O}_{E(q)}) = H^0(\mathcal{L} \otimes \mathcal{O}_{E_0(q)}) \quad \text{and} \quad H^0(p_1(q)^* \mathcal{L}) = H^0(\mathcal{L}).$$

The long exact sequence of cohomologies associated to (4.2) contains

$$0 \rightarrow H^0(\mathcal{M}_q) \rightarrow H^0(\mathcal{L}) \xrightarrow{\rho(q)} H^0(\mathcal{L} \otimes \mathcal{O}_{E_0(q)}),$$

where  $\rho(q)$  is the natural evaluation/restriction map on  $E_0(q)$ . By assumption,  $\rho(q) = 0$  if and only if  $q = p$ . So,  $h^0(\mathcal{M}_q) < h^0(\mathcal{L}) = h^0(\mathcal{M}_p)$  for any  $q \neq p$ .

As  $\mathcal{M}_q$  is the restriction of  $\mathcal{M}$  to  $\overline{Y}_q$ , it follows that  $\gamma$  deforms to  $c_1(\mathcal{M}_q)$  because  $\overline{Y}_p$  deforms to  $\overline{Y}_q$  along the family  $p_2$ . Hence, we have  $\text{NL}(\gamma) = Z$ . By definition,  $\mathcal{B}_{\mathcal{M}_p} \neq Z$ . This completes the proof.  $\square$

**Remark 4.7.** In the proof of Theorem 4.6 it was observed that  $h^0((p_1^*\mathcal{L})|_{\overline{Y}_q}) = h^0(\mathcal{L})$  for any closed point  $q \in Z$ . By Grauert's upper semicontinuity theorem [Har77, Corollary III.12.9], this implies that for any ring homomorphism  $\phi : \mathcal{O}_{Y,p} \rightarrow k(p)[\epsilon]/(\epsilon^2)$ , the homomorphism

$$H^0(p_1^*\mathcal{L}) \otimes k(p)[\epsilon]/(\epsilon^2) \rightarrow H^0(p_1^*\mathcal{L} \otimes_\phi k(p)[\epsilon]/(\epsilon^2))$$

is an isomorphism. Since  $h^0(p_1^*\mathcal{L}) = h^0(p_{1*}p_1^*\mathcal{L}) = h^0(\mathcal{L})$ , this implies that

$$\dim_{k(p)[\epsilon]/(\epsilon^2)} H^0(p_1^*\mathcal{L} \otimes_\phi k(p)[\epsilon]/(\epsilon^2)) = h^0(\mathcal{L}).$$

**Notation 4.8** (Restriction to infinitesimal deformation). Fix a point  $q \in Z$  and a tangent  $t \in T_q Z$  corresponding to a ring homomorphism  $\phi : \mathcal{O}_{Z,q} \rightarrow k(q)[\epsilon]/(\epsilon^2)$ . Denote by  $\overline{Y}_t$  the infinitesimal deformation of  $\overline{Y}_q$  along  $t$ , so  $\overline{Y}_t$  is the fiber product  $\overline{Y} \times_Z \text{Spec } k(q)[\epsilon]/(\epsilon^2)$  with respect to the morphism  $\text{Spec } k(q)[\epsilon]/(\epsilon^2) \rightarrow Z$  induced by  $\phi$ . Given any sheaf  $\mathcal{F}$  on  $\overline{Y}$ , denote by  $\mathcal{F}_t$  the pull-back of  $\mathcal{F}$  to  $\overline{Y}_t$ . In particular,

$$\mathcal{F}_t \cong \mathcal{F} \otimes_\phi k(q)[\epsilon]/(\epsilon^2),$$

where the sheaf  $\mathcal{F} \otimes_\phi k(q)[\epsilon]/(\epsilon^2)$  is defined by

$$(\mathcal{F} \otimes_\phi k(q)[\epsilon]/(\epsilon^2))_y = \mathcal{F}_y \otimes_\phi k(q)[\epsilon]/(\epsilon^2)$$

if  $y \in \overline{Y}_q$  and zero otherwise (consider  $k(q)[\epsilon]/(\epsilon^2)$  as a constant sheaf supported on  $\overline{Y}_q$ ).

**Theorem 4.9.** *There exist  $t \in T_p Z$  and  $D \in H^0(\mathcal{M}_p)$  such that*

$$\rho_\pi(t) \neq 0, \quad t \cup c_1(\mathcal{M}_p) = 0$$

*but  $t \lrcorner \{D\} \neq 0$ . In particular,  $t \in T_p \text{NL}(c_1(\mathcal{M}_p)) = T_p Z$ , but  $t \notin T_p \mathcal{B}_{\mathcal{M}_p}$ .*

*Proof.* Given a morphism  $\phi : \mathcal{O}_{Z,p} \rightarrow k(p)[\epsilon]/(\epsilon^2)$ , the following exact sequence is obtained by tensoring (4.1) with  $-\otimes_\phi k(p)[\epsilon]/(\epsilon^2)$ :

$$0 \rightarrow H^0(\mathcal{M} \otimes_\phi k(p)[\epsilon]/(\epsilon^2)) \rightarrow H^0(p_1^*\mathcal{L} \otimes_\phi k(p)[\epsilon]/(\epsilon^2)) \xrightarrow{\rho(\phi)} H^0(p_1^*\mathcal{L} \otimes \mathcal{O}_E \otimes_\phi k(p)[\epsilon]/(\epsilon^2)).$$

It suffices to show that there exists a morphism  $\phi$  such that  $\rho(\phi)$  is not the zero map. Indeed, given such a  $\phi$ , by Remark 4.7,

$$\dim_{k(p)[\epsilon]/(\epsilon^2)} H^0(\mathcal{M} \otimes_\phi k(p)[\epsilon]/(\epsilon^2)) < \dim_{k(p)[\epsilon]/(\epsilon^2)} H^0(p_1^*\mathcal{L} \otimes_\phi k(p)[\epsilon]/(\epsilon^2)) = h^0(\mathcal{L}).$$

In the proof of Theorem 4.6 it was observed that  $h^0(\mathcal{L}) = h^0(\mathcal{M}_p)$  when  $\rho(p) = 0$ . This implies that for  $t \in T_p Z$  corresponding to  $\phi$ , we have  $\rho(t) \neq 0$  and  $t \notin T_p \mathcal{B}_{\mathcal{M}_p}$ . By Proposition 3.7, there exists  $D \in H^0(\mathcal{M}_p)$  such that  $t \cup c_1(\mathcal{M}_p) = 0$  but  $t \lrcorner \{D\} \neq 0$ . This will prove the theorem.

As  $p$  is a reduced base point of  $H^0(\mathcal{L})$ , there exist  $s \in H^0(\mathcal{L})$ ,  $f_s \in m_p \setminus m_p^2$  and  $g_s \in \mathcal{L}_p$  such that  $s_p = f_s g_s$  and  $g_s \notin m_p \mathcal{L}_p$ ; here  $s_p$  is the image of  $s$  under the localization morphism  $H^0(\mathcal{L}) \rightarrow \mathcal{L}_p$ . By assumption,  $Y$  is smooth. Since  $f_s \in m_p \setminus m_p^2$ , we can choose a regular sequence  $(f_s, f_1, \dots, f_m)$  generating the maximal ideal  $m_p$ . Let

$$\phi : \mathcal{O}_{Z,p} \rightarrow k(p)[\epsilon]/(\epsilon^2)$$

be the ring homomorphism defined by  $1 \mapsto 1$ ,  $f_s \mapsto \epsilon$  and  $f_i \mapsto 0$  for all  $i = 1, \dots, m$ . Then,  $s$  defines a non-zero element

$$s \otimes 1 \in H^0(p_1^* \mathcal{L} \otimes_\phi k(p)[\epsilon]/(\epsilon^2))$$

and the image of  $s \otimes 1 \in H^0(p_1^* \mathcal{L} \otimes_\phi k(p)[\epsilon]/(\epsilon^2))$  under the natural homomorphism

$$\rho(\phi)'_p : H^0(p_1^* \mathcal{L} \otimes_\phi k(p)[\epsilon]/(\epsilon^2)) \longrightarrow H^0(\mathcal{L}_p \otimes_\phi k(p)[\epsilon]/(\epsilon^2))$$

is non-zero.

Now,  $H^0(p_1^* \mathcal{L} \otimes \mathcal{O}_E \otimes_\phi k(p)[\epsilon]/(\epsilon^2)) = H^0(\pi^*(\text{pr}_1^* \mathcal{L}) \otimes \pi^* \mathcal{O}_{E_0} \otimes_\phi k(p)[\epsilon]/(\epsilon^2))$ , which by the projection formula is equal to

$$H^0(\text{pr}_1^* \mathcal{L} \otimes \mathcal{O}_{E_0} \otimes_\phi k(p)[\epsilon]/(\epsilon^2)) = \bigoplus_{q \in B} H^0((\text{pr}_1^* \mathcal{L} \otimes \mathcal{O}_{E_0})_{q \times p} \otimes_\phi k(p)[\epsilon]/(\epsilon^2)).$$

Observe that  $(\text{pr}_1^* \mathcal{L})_{q \times p} \cong \mathcal{L}_q$ . Recall that the composition  $\Delta \hookrightarrow Y \times Y \xrightarrow{\text{pr}_1} Y$  is an isomorphism, hence  $\text{pr}_1^\# : \mathcal{O}_{Y,p} \xrightarrow{\sim} \mathcal{O}_{\Delta,p \times p}$ . Since the only irreducible component of  $E_0$  containing  $p \times p$  is  $\Delta$ , we have

$$\begin{aligned} (\text{pr}_1^* \mathcal{L} \otimes \mathcal{O}_{E_0})_{p \times p} \otimes_\phi k(p)[\epsilon]/(\epsilon^2) &\cong \mathcal{L}_p \otimes_{\text{pr}_1^\#} \mathcal{O}_{\Delta,p \times p} \otimes_\phi k(p)[\epsilon]/(\epsilon^2) \\ &\cong \mathcal{L}_p \otimes_{\text{pr}_1^\#} \mathcal{O}_{Y,p} \otimes_\phi k(p)[\epsilon]/(\epsilon^2) \cong \mathcal{L}_p \otimes_\phi k(p)[\epsilon]/(\epsilon^2), \end{aligned}$$

where  $M \otimes_{\text{pr}_1^\#} N$  is a tensor product of  $\mathcal{O}_{Y \times Z, p \times p}$ -modules viewed as  $\mathcal{O}_{Y,p}$ -modules under the morphism  $\text{pr}_1^\#$ . Write the evaluation map  $\rho(\phi) = \bigoplus_{q \in B} \rho(\phi)_q$ , where  $\rho(\phi)_q$  is the restriction of the evaluation map to  $q$ . Then  $\rho(\phi)_p$  coincides with the morphism  $\rho(\phi)'_p$  defined above. Since  $\rho(\phi)'_p$  is non-zero, so is  $\rho(\phi)$ . This completes the proof of the theorem.  $\square$

## 5. APPLICATIONS TO CURVE COUNTING

**Setup 5.1.** Let  $Y$  be a smooth projective surface and  $\mathcal{L}$  an invertible sheaf on  $Y$ . Denote by  $r := h^0(\mathcal{L})$ . Fix  $m$  distinct points  $p_1, \dots, p_m$  on  $Y$ . Define

$$W := \{s \in H^0(\mathcal{L}) \mid s(p_i) = 0 \ \forall i = 1, \dots, m\}$$

and

$$W_q := \{s \in H^0(\mathcal{L}) \mid s(p_i) = 0 \ \forall i = 2, \dots, m, \text{ and } s(q) = 0\}.$$

For  $r > 0$ , define

$$\mathcal{Z}_{\mathcal{L}}^r(p_1, \dots, p_m) := \{q \in Y \setminus \{p_2, p_3, \dots, p_m\} \mid \dim W_q \geq r\}.$$

Observe that  $\mathcal{Z}_{\mathcal{L}}^r(p_1, \dots, p_m)$  is the set of points  $q \in Y \setminus \{p_2, \dots, p_m\}$  such that there exists at least an  $r$  dimensional family of curves in the linear system  $|\mathcal{L}|$  which passes through the points  $q, p_2, p_3, \dots, p_m$ . We will prove that the locus of such points coincides with the Brill-Noether type locus defined in the previous section.

**Theorem 5.2.** *Notations as in Section 4. Substitute  $p = p_1$  and  $B = \{p_1, \dots, p_m\}$ . For  $r = \dim W$ , we have  $\mathcal{Z}_{\mathcal{L}}^r(p_1, \dots, p_m) = \mathcal{B}_{\mathcal{M}_{p_1}}$ .*

*Proof.* By the proof of Theorem 4.6, there is a short exact sequence

$$0 \longrightarrow H^0(\mathcal{M}_q) \longrightarrow H^0(\mathcal{L}) \xrightarrow{\rho(q)} H^0(\mathcal{L} \otimes \mathcal{O}_{E_0(q)}).$$

Recall that  $E_0(q) = B_p \cup q$  and  $\rho(q)$  is the evaluation at  $E_0(q)$ . Then by definition,  $\ker \rho(q) = W_q$ . Hence,  $H^0(\mathcal{M}_q) = W_q$  and  $H^0(\mathcal{M}_{p_1}) = W$ . The theorem now follows directly.  $\square$

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